Lecture 31: Power series representations of Functions

From our knowledge of Geometric Series, we know that

$$g(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1.$$

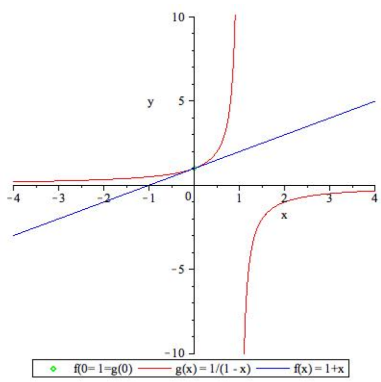
(Recall that we had

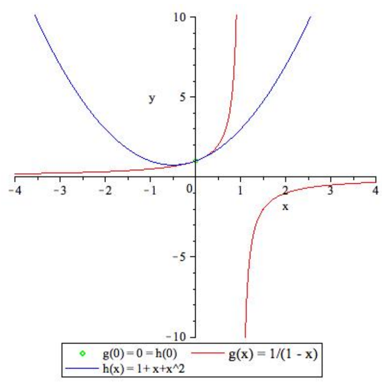
$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if} \quad -1 < r < 1$$

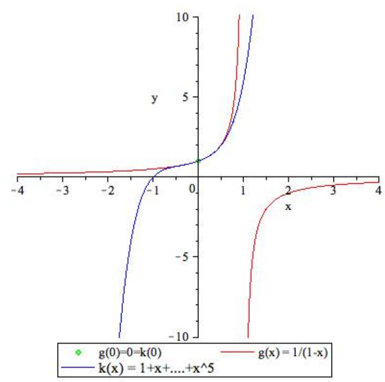
and this series diverges for $|r| \geq 1$.
Above we have $a = 1$ and $x = r$.)

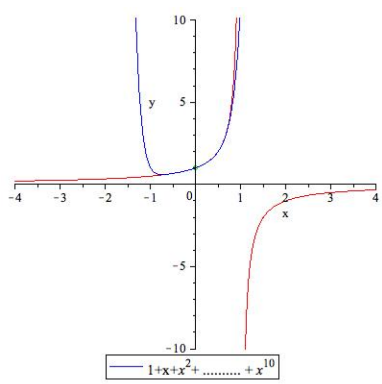
This gives us **a power series representation for the function $g(x)$ on the interval $(-1, 1)$.**
Note that the function $g(x)$ here has a larger domain than the power series.

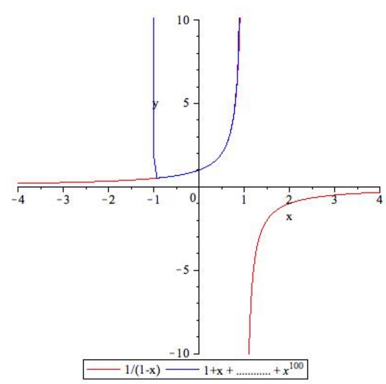
The n th partial sum of the above power series is given by $P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.
Hence, as $n \to \infty$, the graphs of the polynomials, $P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$ get closer to the graph of $f(x)$ on the interval $(-1, 1)$.











Having a power series representation of a function on an interval is useful for the purposes of integration, differentiation and solving differential equations.

Method of Substitution

First, we examine how to use the power series representation of the function $g(x) = 1/(1-x)$ on the interval $(-1, 1)$ to derive a power series representation of other functions on an interval.

Example (Substitution) Find a power series representation of the functions given below and find the interval of convergence of the series.

$$f(x) = \frac{1}{1 + x^7},$$

Example (Substitution) Find a power series representation of the functions given below and find the interval of convergence of the series.

$$f(x) = \frac{2x}{1 + x},$$

Example (Substitution) Find a power series representation of the functions given below and find the interval of convergence of the series

$$h(x) = \frac{1}{4-x}$$

Differentiation and Integration of Power Series

We can differentiate and integrate power series term by term, just as we do with polynomials.

Theorem If the series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

Also

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of both of these power series is also R. (The interval of convergence may not remain the same when a series is differentiated or integrated; in particular convergence or divergence may change at the end points).

Example Find a power series representation of the function

$$\frac{1}{(x+1)^2}.$$

Example (Integration) Find a power series representation of the function

$$\ln(1+x).$$

Extra (Summing Series) Use the **fact that a power series (with x values in the real numbers) is continuous on its domain** to show that

$$\sum_{x=0}^{\infty} \frac{(-1)^n}{n+1} = \ln(2).$$

We have

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for} \quad -1 < x < 1.$$

When $x = -1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{1}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges, by comparison with the harmonic series.
When $x = 1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ which converges , by the alternating series test.
Therefore the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is $(-1, 1]$ and since this power series is continuous on its interval of convergence, we have

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

. Using the fact derived above that $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \ln(1+x)$ for $x < 1$, we have

$$\lim_{x \to 1^-} \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

or

$$\ln(2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}.$$

Example (Approximation) Use power series to approximate the following integral up to 4 decimal places:

$$\int_0^{0.1} \frac{1}{1+x^7}dx$$

Extra Example (Integration) Show that

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}dx$$

and use your answer calculate $\sum_{n=0}^{\infty}(-1)^n\frac{1}{\sqrt{3}^{(2n+1)}(2n+1)}$.

Using the power series representation of $\frac{1}{1-x}$ on the interval $(-1,1)$, we get a power series representation of $\frac{1}{1+x^2}$:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots \quad \text{on the interval} \quad (-1,1).$$

Now we can integrate term by term to get a power series representation of $\tan^{-1}(x)$ on the interval $(-1,1)$,

$$\tan^{-1}(x) = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots \quad \text{on the interval} \quad (-1,1).$$

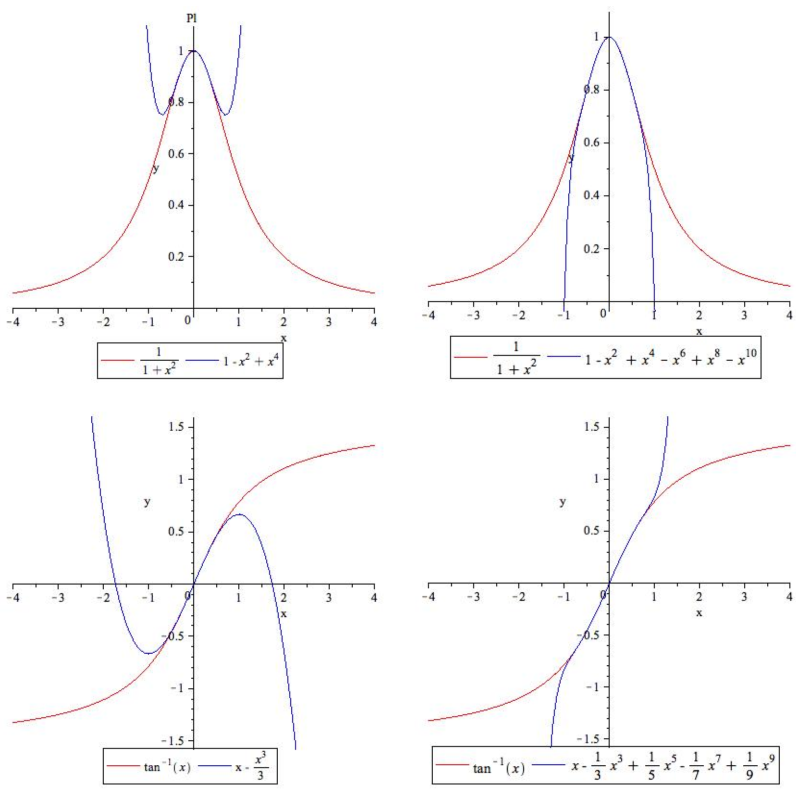
Since $\tan^{-1}(0) = 0$, we have $C = 0$ and

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots \quad \text{on the interval} \quad (-1,1).$$

Since $\frac{1}{\sqrt{3}} < 1$ and with $x = \frac{1}{\sqrt{3}}$, we get

$$\frac{\pi}{6} = \tan^{-1}(\frac{1}{\sqrt{3}}) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{\sqrt{3}^{(2n+1)}(2n+1)}.$$

The pictures shown below are of the 4th and 10th partial sums of the above two series, alongside the graphs of the corresponding functions.



Extra Example:(Substitution) Find a power series representation of the function

$$\tan^{-1}\left(\frac{x}{2}\right)$$

We saw above that

$$\tan^{-1}(y) = \sum_{n=0}^{\infty}(-1)^n \frac{y^{2n+1}}{2n+1} = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} - \cdots \quad \text{on the interval} \quad (-1, 1).$$

Letting $y = \frac{x}{2}$, we get

$$\tan^{-1}(\frac{x}{2}) = \sum_{n=0}^{\infty}(-1)^n \frac{(x/2)^{2n+1}}{2n+1} = (x/2) - \frac{(x/2)^3}{3} + \frac{(x/2)^5}{5} - \frac{(x/2)^7}{7} - \cdots \quad \text{for} \quad -1 < \frac{x}{2} < 1.$$

giving us that

$$\tan^{-1}\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot 2^{2n+1}} = \left[\frac{x}{2} - \frac{x^3}{3\cdot 2^3} + \frac{x^5}{5\cdot 2^5} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot 2^{2n+1}} + \cdots\right]$$

for $-2 < x < 2$.

Extra Example; Two methods, same result

Find a power series representation of the function

$$\ln\left(x^2+4\right)$$

First Way: Substitution and Integration We have

$$\ln(x^2+4) = \int \frac{2x}{x^2+4}\,dx$$

Check that

$$\frac{1}{4+x^2} = \left[\frac{1}{4} - \frac{x^2}{4^2} + \frac{x^4}{4^3} - \frac{x^6}{4^4} + \cdots + \frac{(-1)^n x^{2n}}{4^{n+1}} + \cdots\right] = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}\right] \quad \text{for} \quad -2 < x < 2$$

Hence

$$\frac{2x}{4+x^2} = \left[\frac{2x}{4} - \frac{2x\cdot x^2}{4^2} + \frac{2x\cdot x^4}{4^3} - \frac{2x\cdot x^6}{4^4} + \cdots + \frac{(-1)^n 2x\cdot x^{2n}}{4^{n+1}} + \cdots\right] = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x\cdot x^{2n}}{4^{n+1}}\right] \quad \text{for} \quad -2 < x < 2$$

$$\left[\frac{2x}{4} - \frac{2x^3}{4^2} + \frac{2x^5}{4^3} - \frac{2x^7}{4^4} + \cdots + \frac{(-1)^n 2x^{2n+1}}{4^{n+1}} + \cdots\right] = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+1}}{4^{n+1}}\right] \quad \text{for} \quad -2 < x < 2$$

Integrating both sides, we get

$$\ln(x^2+4) = \int \frac{2x}{x^2+4}\,dx = C + \left[\int \frac{2x}{4}\,dx - \int \frac{2x^3}{4^2}\,dx + \int \frac{2x^5}{4^3}\,dx - \int \frac{2x^7}{4^4}\,dx + \cdots + \int \frac{(-1)^n 2x^{2n+1}}{4^{n+1}}\,dx + \cdots\right]$$

$$= C + \left[\sum_{n=0}^{\infty} \int \frac{(-1)^n 2x^{2n+1}}{4^{n+1}}\,dx\right] \quad \text{for} \quad -2 < x < 2.$$

Substituting $x = 0$ into the equation, we get $\ln 4 = C$. Thus we get

$$\ln(x^2+4) = \ln 4 + \left[\frac{2x^2}{2\cdot 4} - \frac{2x^4}{4\cdot 4^2} + \frac{2x^6}{6\cdot 4^3} - \frac{2x^8}{8\cdot 4^4} + \cdots + \frac{(-1)^n 2x^{2n+2}}{(2n+2)\cdot 4^{n+1}} + \cdots\right]$$

$$= \ln 4 + \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+2}}{(2n+2)\cdot 4^{n+1}}\right] \quad \text{for} \quad -2 < x < 2$$

$$= \ln 4 + \left[\frac{x^2}{4} - \frac{x^4}{2\cdot 4^2} + \frac{x^6}{3\cdot 4^3} - \frac{x^8}{4\cdot 4^4} + \cdots + \frac{(-1)^n x^{2n+2}}{(n+1)\cdot 4^{n+1}} + \cdots\right] = \boxed{\ln 4 + \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(n+1)\cdot 4^{n+1}}\right]} \quad \text{for} \quad -2 < x < 2$$

Second Way (Substitution): $\ln(4+x^2) = \ln(4(1+\frac{x^2}{4}) = \ln(4) + \ln(1+\frac{x^2}{4})$. We can now use our result from before

$$\ln(1+y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{n+1}}{n+1}.$$

Let $y = \frac{x^2}{4}$ to get

$$\ln(1+\frac{x^2}{4}) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2/4)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^{n+1}(n+1)}$$

Hence

$$\boxed{\ln(4+x^2) = \ln(4) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^{n+1}(n+1)}.}$$

Extra Example: (Substitution) Find a power series representation of the functions given below and find the interval of convergence of the series.

$$g(x) = \frac{2x^2}{3-x},$$

We have

$$\frac{2x^2}{3-x} = \frac{2x^2}{3}\left[\frac{1}{1-(x/3)}\right].$$

Now recall from above that

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots + y^n + \cdots = \sum_{n=0}^{\infty} y^n \quad \text{for} \quad -1 < y < 1$$

Therefore, substituting $x/3$ for y, we get

$$\frac{1}{1-\left(\frac{x}{3}\right)} = 1 + \left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots + \left(\frac{x}{3}\right)^n + \cdots = \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \quad \text{for} \quad -1 < \left(\frac{x}{3}\right) < 1$$

We have $-1 < \left(\frac{x}{3}\right) < 1$ if $-3 < x < 3$ (multiplying the inequality by 3). Therefore

$$\frac{1}{1-\left(\frac{x}{3}\right)} = 1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots + \frac{x^n}{3^n} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{3^n} \quad \text{for} \quad -3 < x < 3.$$

Now we want a power series representation for

$$g(x) = \frac{2x^2}{3-x} = \frac{2x^2}{3}\left[\frac{1}{1-(x/3)}\right]$$

using the power series derived above for $\frac{1}{1-(x/3)}$, we get

$$\frac{2x^2}{3-x} = \frac{2x^2}{3}\left[1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots + \frac{x^n}{3^n} + \cdots\right] = \frac{2x^2}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n} \quad \text{for} \quad -3 < x < 3.$$

or

$$\frac{2x^2}{3-x} = \left[\frac{2x^2}{3} + \frac{2x^2}{3}\left(\frac{x}{3}\right) + \frac{2x^2}{3}\left(\frac{x^2}{3^2}\right) + \frac{2x^2}{3}\left(\frac{x^3}{3^3}\right) + \cdots + \frac{2x^2}{3}\left(\frac{x^n}{3^n}\right) + \cdots\right] = \sum_{n=0}^{\infty}\frac{2x^2}{3}\left(\frac{x^n}{3^n}\right) \quad \text{for} \quad -3 < x < 3.$$

or

$$\frac{2x^2}{3-x} = \left[\frac{2x^2}{3} + \frac{2x^3}{3^2} + \frac{2x^4}{3^3} + \frac{2x^5}{3^4} + \cdots + \frac{2x^{n+2}}{3^{n+1}} + \cdots\right] = \sum_{n=0}^{\infty}\frac{2x^{n+2}}{3^{n+1}} \quad \text{for} \quad -3 < x < 3.$$